Computational Complexities of Turing Machine

Prof. (Dr.) K.R. Chowdhary
Email: kr.chowdhary@iitj.ac.in

Formerly at department of Computer Science and Engineering
MBM Engineering College, Jodhpur

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Resources requirements for TM Computations

- Complexity is *Rate of growth* of computation in time and space.
- DTM: The time complexity is measured in terms of number of transitions in a computation, and space complexity is maximum number of tape squares used.
- Multi-tape TM: Complexity differs polynomially from standard TM
- Computations of a computer can be simulated on a TM, and it can be shown that no. of transitions in TM = polynomial expression $\times$ no. of instructions executed in computer.
- $\therefore$, resource bounds in TM provide practical information about complexity of algorithms and computer programs.

  Complexity:

  Objective: 1) Assessment of Algorithms of problems, 2) Comparison of difficulty of problems.

  Problem Examples:

  1. Array sort: Input: $A_1 \ldots A_n$, Output: $A_{1'} \ldots A_{n'}$
  2. Squaring a Matrix: Input: $B : n \times n$, output: $C = B^2$
  3. Path problem for directed graph: Input: $G = (V, E)$, Output: Yes, if there is a path from v_i to v_j, else No.

Finding Complexity

4 Acceptance by a TM that halts for all inputs: Input: String w, output: Yes, if M accepts w, No otherwise

- Case 1,2: Computes function, case 3,4: are decision problems. Resource usage is measured in: case 1 as no. of data movements, in case 2 as number of scalar multiplications, in 3 as No. of nodes visited, 4 as No. of transitions in a computation. There is no definite relation in complexities in above.
- Church-Turing thesis says that TM is an effective computing machine, $\therefore$ time and space complexities of TMs describe number of transitions and & number of squares on tape for all the problems.
- Rate of growth:

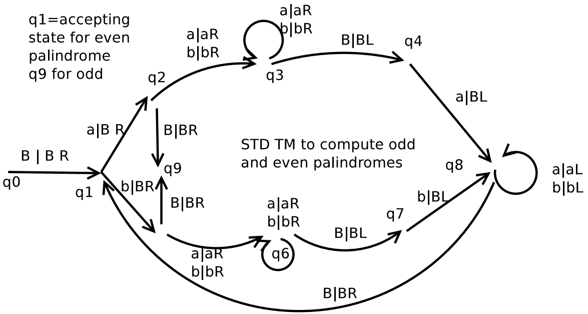
  Let $f(n) = n^2$, and function $g(n) = n^3$.

  $\therefore f \in O(g)$

  $g \notin O(f)$, $\therefore$, $n^2 \in O(n^3)$, since $n^2 \leq_P n^3$ (n^2 is polynomially reducible to n^3)

Time Complexity of Standard TM

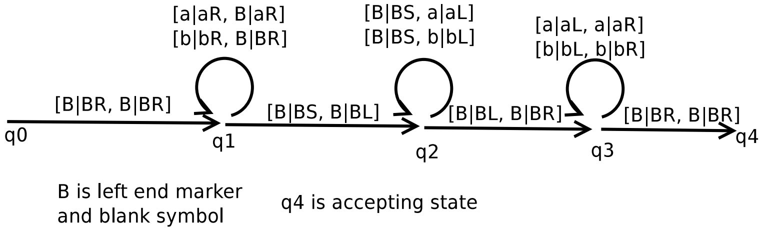
- TC (Time complexity) is measured in terms of number of transitions, for TM M for input $|w| = n$
- Example: Analyze the computation for palindrome problem over $\Sigma = \{a, b\}$ for TM



- Let $tc_m(n)$ is time complexity. Thus, $tc_m(0) = 1$, $tc_m(1) = 3$, $tc_m(2) = 6$, $tc_m(3) = 10$,
$\therefore tc_m(n) = \Sigma_{i=1}^{n+1} i = (n+2)(n+1)/2 = O(n^2)$

Time Complexity of Two tape TM

- Exercise: Time complexity for palindrome problem over $\Sigma = \{a, b\}$



- Working: Loop1= Copy, Loop2 = Rewind tape2, Loop3 = Compare tape1 and tape2.
- Accepting computation: 3 passes, $tc_m(n) = 3(n+1) = 1$, is polynomial.

  $\therefore$ complexity of standard TM = polynomial $\times$ time complexity of 2-tape TM.

Complexity in TM variations

Theorem

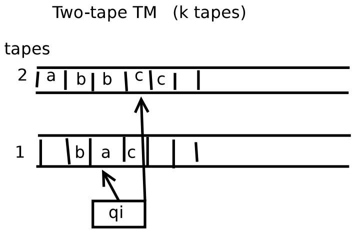
Let L be language accepted by K-track DTM M with Time complexity $tc_M(n)$, then L is also accepted by standard TM M' with T.C. $tc_{M'}$, then $tc_M(n) = tc_{M'}(n)$.

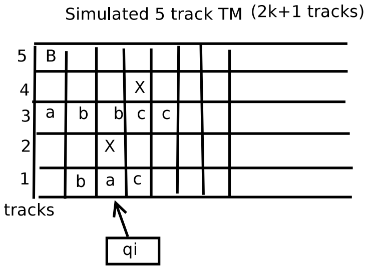
Proof.

- *In k-track TM $\delta(q_i, x_1, \ldots, x_k)$, where x_i is symbol in track i.*
- *The same transition in std-TM (one tape) is $\delta(q_i, [x_1, \ldots, x_k])$, where k-tuple is single alphabet in std-TM M'. $\therefore$, $tc_M = tc_{M'}$.*

□

- K tape TM can be simulated by a $2k+1$ track TM





- In Two tape:Two heads, Multi-track: one head. The Symbol X in multi-track indicates virtual head position of two tape TM. To simulate a transition of two-tape, the 5 track, the TM locates a symbol at corresponding virtual head (X), reads the symbol under that, and performs transition as that in two tape TM. Obviously, the complexity of multi-track is more than that of k-tape TM.

Time Complexity in TM variations

Theorem

Let L be language accepted by K-tape DTM M with Time complexity $tc_M(n) = f(n)$, then L is also accepted by standard TM N with time complexity $tc_N(n)$, where $tc_N(n) \in O(f(n)^2)$.

Proof.

- *We will first simulate K-tape TM on multi-track TM M'. The no. of tracks will be 2k+1, as proved earlier.*
- *$\therefore$, we should show only: $tc_{M'} \in (f(n)^2)$. (Because complexities of multi-track and standard TM are equal).*
- *Let us try to simulate the farthest t^{th} transition of M on M'.*

  1. There are k-even tracks for each head position symbol X. There are $2 \times t$ transitions. Total transitions $= 2(t+1)$, one extra for head position, on $2t+1^{th}$ track. Contd . . .

$\square$

Time Complexity in TM variations . . .

2. Similarly, for reading-writing of symbols on all odd tracks, transitions are 2t.

3. For k tape TM M, total transitions = $k(2t+2(t+1))+1 = 4kt+2k+1$. Thus $tc_{M'}(n) \leq 1+\Sigma_{t=1}^{f(n)}(4kt+2k+1) \in (f(n)^2)$.

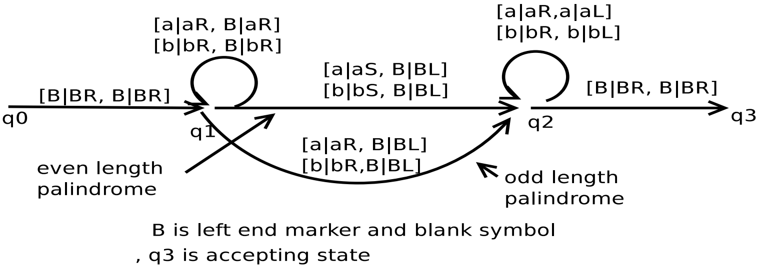
- Since, time complexities of Std TM and multi-track are same, the above time complexity is of std TM. □

Time Complexity of Nondeterministic TM

- A DTM generates and examins all possibilities
- A NDTM employs a Guess-and-check strategy, and needs to determine if one of the possibilities provide the solution
- **Example**: To find out if k is composite number, we try to divide it by all the possible factors of it. If one exists, then it is composite. The set of all the composite numbers is language COMPOSITS.
- A NDTM chooses a guess value, and a single division (verification) determines that it is factor.
- A string is accepted by NDTM if at least one of the computation terminates in accepting state.

Two-tape NDTM to decide a palindrome language

- Let $\Sigma = \{a, b\}$. The palindrome may be odd or even.



- Operations:

1. Read and copy 1st half from tape1 to tape2,
2. nondeterministically decide the centre, at tape 1,
3. compare 2nd half on tape 1 with first half on tape 2.

- No. of transitions for input $|w| = n$, is n+2 for odd n, N+3 for even n. $tc_M(n) = O(n)$.

Simulation of NDTM by DTM

Theorem

Let L be language accepted by NDTM M with $tc_M(n) = f(n)$, for input $|w| = n$. Then, L is accepted by DTM M' with T.C. $tc_{M'} = O(f(n)c^{f(n)})$, where c is maximum number of transitions for any State $\times$ Symbol pair of M.

Proof.

- *Let $M = (Q, \Sigma, \Gamma, \delta, q_0, H)$, that halts for all the inputs. Transformation from M to M' is done as follows:*
- *A unique computation of M has sequence $(m_1, \ldots, m_n), 1 \leq m_i \leq c$; m_i is which of the c possible transformations are executed in ith step. For $|w| = n$, maximum transitions of M is f(n). Contd next ...*

□

Simulation of NDTM by DTM ...

- For one computation of M, the M'
  1. generates sequence $m_1, \ldots, m_n$
  2. Simulate M for this sequence
  3. if solution not found, goto step a.
- In worst case no. of transitions in M' is $c^{f(n)}$, as so many sequences are to be analyzed.
- T.C. of M' is $O(f(n)c^{f(n)})$. $\square$

Note: This is similar to a bianry tree where the lowest level there are 2^h node, where h is is responsibility of .