Discrete Structures and Set Operations

Set Theory Essentials

Prof. K R Chowdhary

Former Prof & Head, Dept. of Computer Sc. & Engineering
MBM Engineering College (Now MBM University)

Why Discrete Structures?

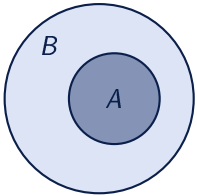
- Mathematical reasoning: logic, proofs, induction
- Discrete structures: sets, relations, graphs, trees, finite state machines
- Combinatorial analysis: counting and enumeration
- Modeling and algorithms: proving correctness, estimating time and memory

Key idea

Computers store and manipulate information discretely, so discrete mathematics is their natural foundation.

Sets, Members and Subsets

- A set is a collection of distinct objects; order and repetition do not matter: $\{a, b, c\} = \{c, b, a, a\}$
- Membership: $a \in S$, non-membership: $d \notin S$
- Set-builder notation: $S = \{x \mid P(x)\}$
- Empty set $\varnothing = \{\}$ is a subset of every set
- $P \subseteq Q$; proper subset $P \subset Q$; always $S \subseteq S$
- Equal sets can look different: $\{2, 4, 6, 8, 10\} = \{x \mid x \text{ even}, 2 \leq x \leq 10\}$

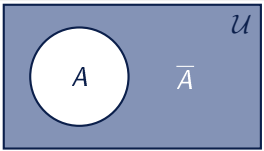


$A \subset B$: every element of A is also in B

Universe, Complement and Ordered Pairs

Complement

With $\mathcal{U} = \mathbb{Z}$ and $A = \{2, 3, 5, 7\}$:
$\overline{A} = \{x \in \mathcal{U} \mid x \notin A\}$



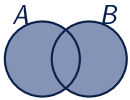
Ordered pair (Def. 1.1)

$(a, b) = \{\{a\}, \{a, b\}\}$

Why this works: the singleton $\{a\}$ marks the *first* coordinate, and the other element of $\{a, b\}$ is the second.

Theorem 1.1: $(a, b) = (a', b')$ iff $a = a'$ and $b = b'$. Hence $(a, b) \neq (b, a)$ when $a \neq b$.

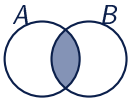
Core Set Operations in Pictures



Union

$A \cup B$

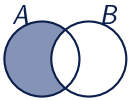
in A or B



Intersection

$A \cap B$

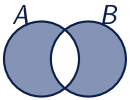
in A and B



Difference

$A - B$

in A, not B



Symmetric diff.

$A \oplus B$

in exactly one

Example: $A = \{a, b, c, d\}$, $B = \{c, d, f, g\}$

$A \cap B = \{c, d\}, \quad A - B = \{a, b\}, \quad A \oplus B = (A \cup B) - (A \cap B) = \{a, b, f, g\}$

Disjoint sets have $A \cap B = \varnothing$. The power set $P(\{a, b, c\})$ has $2^3 = 8$ elements.

Laws of Sets and Proof Technique

- Commutative, associative, idempotent
- Identity: $A \cup \varnothing = A,\ A \cap \mathcal{U} = A$
- Inverse: $A \cup \overline{A} = \mathcal{U},\ A \cap \overline{A} = \varnothing$
- Domination: $A \cup \mathcal{U} = \mathcal{U},\ A \cap \varnothing = \varnothing$
- Absorption: $A \cup (A \cap B) = A$

Distributive laws (Thm 1.3)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Method (double inclusion): take $x \in A \cap (B \cup C)$. Then $x \in A$ and ($x \in B$ or $x \in C$), so $x \in A \cap B$ or $x \in A \cap C$. Reverse the argument for $\supseteq$.

De Morgan's Laws

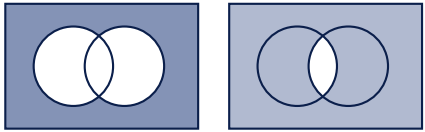
Theorem 1.4

$\overline{A \cap B} = \overline{A} \cup \overline{B}$ and $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Proof of the second law:

$$\begin{aligned} x \in \overline{A \cup B} &\iff x \notin A \cup B \\ &\iff x \notin A \text{ and } x \notin B \\ &\iff x \in \overline{A} \cap \overline{B} \end{aligned}$$

Uses: circuit design (NAND/NOR), negation normal form, probability.



Left: outside both circles. Right: overlap of "outside A" and "outside B" (darkest). Same region.

Using the Laws: Worked Identities

Theorem 1.5: $(A \cap B) - (A \cap C) = A \cap B \cap C'$

$$
\begin{aligned}
(A \cap B) - (A \cap C) &= (A \cap B) \cap (A \cap C)' && X - Y = X \cap Y' \\
&= (A \cap B) \cap (A' \cup C') && \text{De Morgan} \\
&= (A \cap B \cap A') \cup (A \cap B \cap C') && \text{distributive} \\
&= \varnothing \cup (A \cap B \cap C') = A \cap B \cap C' && A \cap A' = \varnothing
\end{aligned}
$$

Theorem 1.6: $(A \cup B) - (C - A) = A \cup (B - C)$

$$
\begin{aligned}
(A \cup B) - (C - A) &= (A \cup B) \cap (C \cap A')' && X - Y = X \cap Y' \\
&= (A \cup B) \cap (C' \cup A) && \text{De Morgan} \\
&= A \cup (B \cap C') = A \cup (B - C) && \text{distributive}
\end{aligned}
$$

Finite, Countable and Uncountable Sets

- Same cardinality $\iff$ a one-to-one correspondence exists
- Countably infinite: matches $\mathbb{N}$; evens via $i \mapsto 2i$, odds via $i \mapsto 2i+1$
- Infinite subsets and unions of two countable sets stay countable (Thm 1.7, 1.8)
- Cantor (Thm 1.9): $\mathbb{R}$ is uncountable

Diagonal argument. Suppose all reals in $(0,1)$ are listed:

$$\begin{aligned} f(0) &= 0.\,\mathbf{3}\,1\,4\,1\cdots \\ f(1) &= 0.\,7\,\mathbf{1}\,8\,2\cdots \\ f(2) &= 0.\,5\,0\,\mathbf{0}\,0\cdots \\ f(3) &= 0.\,3\,3\,3\,\mathbf{3}\cdots \\ \hline x &= 0.\,6\,8\,9\,6\cdots \end{aligned}$$

Set $b_i = 9 - a_{ii}$ (or 0 if $a_{ii} = 9$). Then x differs from every $f(i)$ at digit i, so it is not in the list.

Russell's Paradox and Takeaways

Russell's paradox

Let $R = \{x \mid x \notin x\}$.

- If $R \in R$, then by definition $R \notin R$.
- If $R \notin R$, then by definition $R \in R$.

Contradiction either way.

Theorem 1.10: there is no set of all sets, so set formation must be restricted.

Takeaways

- Sets are the foundation of mathematics
- Operations obey a compact algebra of laws
- Prove equality by double inclusion
- Infinity comes in different sizes

Reference: K. R. Chowdhary, *Fundamentals of Discrete Mathematical Structures*, 3rd ed., PHI Learning, 2015 [Contents].

Relevant chapters: Discrete Structures and Set Theory; Relations; Mathematical Logic.