Propositional Logic and Logical Inference

Logic Essentials

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Why Logic?

- Hardware design: gates and circuits are propositional formulas realised in silicon
- Verification: truth tables and inference check hardware, software and natural-language arguments
- Predicate logic: adds variables; basis of high-level programming languages and complex arguments
- Proofs: a sequence of statements (hypotheses, axioms, derived conclusions) using rules of inference
- Scale problem: a truth table for n variables has 2^n rows ($n = 10 \Rightarrow 1024$), so inference rules are preferred

Key idea

A computation is like a proof: every step is elementary, justified, and nothing is left to intuition.

Propositions and Well-Formed Formulas

- A proposition is an atomic statement, true or false
- Symbols: $\wedge$ and, $\vee$ or, $\neg$ not, $\rightarrow$ implies, $\leftrightarrow$ iff, $\perp$ contradiction
- Parentheses give unique readability: $(p \wedge q) \rightarrow r \neq p \wedge (q \rightarrow r)$

p: Rama is a second-year student
q: Rama plays tennis
$\neg p \rightarrow q$: if Rama is not a second-year student, then Rama plays tennis

Recursive definition of a wff

1. Every propositional variable is a formula
2. If p, q are formulas, so are $\neg p,\ p \wedge q,\ p \vee q,\ p \rightarrow q,\ (p)$
3. Nothing else is a formula

BNF

formula ::= atom | ¬formula | formula∨formula | formula∧formula | (formula) | formula→formula
atom ::= $\perp$ | p | q | r | p_1 | ...

Semantics and Truth Tables

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Reading $p \rightarrow q$

$p =$ "you work hard", $q =$ "you succeed". The statement is false only when p is true and q is false; if p is false, nothing is violated, so it is true.

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

- Tautology $\models u$: true in every row Contradiction: false in every row
- Satisfiable: true in at least one row; u is a contradiction iff $\neg u$ is a tautology

Logical Equivalences

De Morgan	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
Implication	$p \rightarrow q \Leftrightarrow \neg p \vee q$
Distributive	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
Identity	$p \wedge T \Leftrightarrow p, \quad p \vee F \Leftrightarrow p$
Inverse	$p \wedge \neg p \Leftrightarrow F, \quad p \vee \neg p \Leftrightarrow T$
Domination	$p \vee T \Leftrightarrow T, \quad p \wedge F \Leftrightarrow F$
Idempotent	$p \vee p \Leftrightarrow p, \quad p \wedge p \Leftrightarrow p$
Commut./Assoc.	for both $\wedge$ and $\vee$

Equivalence

Two formulas are logically equivalent when their truth tables are identical, i.e. each is a logical consequence of the other.

Converse and contrapositive

For $p \rightarrow q$:
converse: $q \rightarrow p$
contrapositive: $\neg q \rightarrow \neg p$
The contrapositive is equivalent to $p \rightarrow q$; the converse is not.

Simplifying with the Laws

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

	$(p \wedge q) \rightarrow (p \vee q)$	
$\Leftrightarrow$	$\neg(p \wedge q) \vee (p \vee q)$	implication law
$\Leftrightarrow$	$(\neg p \vee \neg q) \vee (p \vee q)$	De Morgan
$\Leftrightarrow$	$(\neg p \vee p) \vee (\neg q \vee q)$	commutative, associative
$\Leftrightarrow$	$T \vee T$	inverse law
$\Leftrightarrow$	T	idempotent

Why simplify instead of a table?

A table needs 2^n rows, while a few law applications give the same answer symbolically and scale to many variables.

Normal Forms: DNF and CNF

- DNF: OR of elementary products (*minterms*)
- CNF: AND of elementary sums (*maxterms*)
- Principal forms: every variable (or its complement) appears in every term

p	q	$p \to q$	$p \vee q$
F	F	T	F
T	F	F	T
F	T	T	T
T	T	T	T

$p \to q \Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q)$

$p \to q \Leftrightarrow (\neg p \vee q)$ (CNF)

DNF $\to$ CNF (three variables)

Start: $\neg a \neg bc + \neg ab \neg c + \neg abc + a \neg bc$

1. List all $2^3 = 8$ combinations
2. Cross out those present in the DNF
3. Negate each variable in the rest
4. OR within terms, AND the terms

$(a+b+c)(\neg a+b+c)(\neg a+\neg b+c)(\neg a+\neg b+\neg c)$

The same method converts CNF to DNF; a truth table confirms equality.

Rules of Inference

Rule	Form	Reads
Modus ponens	$[p \land (p \to q)] \to q$	$p,\ p \to q\ \therefore\ q$
Modus tollens	$[(p \to q) \land \neg q] \to \neg p$	$p \to q,\ \neg q\ \therefore\ \neg p$
Syllogism	$[(p \to q) \land (q \to r)] \to (p \to r)$	chain implications
Disjunctive syllogism	$[(p \lor q) \land \neg p] \to q$	$p \lor q,\ \neg p\ \therefore\ q$
Conjunction	$p,\ q\ \therefore\ p \land q$	combine facts
Simplification	$(p \land q) \to p$	drop a conjunct
Addition	$p \to (p \lor q)$	weaken to a disjunction
Proof by cases	$[(p \to r) \land (q \to r)] \to [(p \lor q) \to r]$	both cases give r
Contradiction	$(\neg p \to \bot) \to p$	refute $\neg p$

Each rule is a tautology, so it can be verified by a truth table once and reused.

A Worked Argument

Argument

If the auditorium was not available or there were examinations, the programme was postponed. If postponed, a new date was announced. No new date was announced. **Therefore, the auditorium was available.**

p: auditorium available
q: there were examinations
r: programme postponed
s: new date announced

1.	$(\neg p \vee q) \rightarrow r$	premise
2.	$r \rightarrow s$	premise
3.	$\neg s$	premise
4.	$\neg r$	modus tollens, 2, 3
5.	$\neg(\neg p \vee q)$	modus tollens, 1, 4
6.	$p \wedge \neg q$	De Morgan, 5
7.	$\therefore\ p$	simplification, 6

The conclusion follows from the premises, so the argument is valid.

Deduction, Soundness and Further Reading

Deductive proof $\Sigma \vdash \alpha$

Each line is an axiom, a member of Σ, or follows by modus ponens $\{\varphi, \varphi \to \psi\} \vdash \psi$.

A1: $p \to (q \to p)$
A2: $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
A3: $(\neg q \to \neg p) \to ((\neg q \to p) \to q)$

Soundness and completeness

Sound: $\Sigma \vdash \theta \;\Rightarrow\; \Sigma \models \theta$
Complete: $\Sigma \models \theta \;\Rightarrow\; \Sigma \vdash \theta$
Hence $\Sigma \vdash \theta \iff \Sigma \models \theta$.

Reference

K. R. Chowdhary, *Fundamentals of Discrete Mathematical Structures*, 3rd ed., PHI Learning, 2015. [Contents]

- Mathematical Logic: truth tables, equivalences, normal forms
- Logical Inferencing: rules of inference, deduction, soundness and completeness
- Predicate Logic: adds variables and quantifiers
- Discrete Structures and Set Theory: set laws mirror logical equivalences