Predicate Logic and Formal Systems

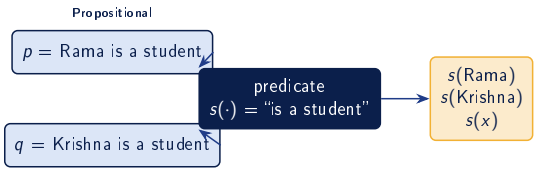
Fundamentals of Discrete Mathematical Structures

Based on the book by Prof. K.R. Chowdhary

$\forall x \, \exists y \; \ldots \quad \Box\Diamond \quad \sigma(n)$

1. Why Predicate Logic?

- Propositional logic treats every atomic statement as opaque.
- Two statements with the same structure get unrelated symbols.
- Predicate logic exposes the common **predicate** and its **arguments**.
- A variable generalises: "x is a student" $= s(x)$.



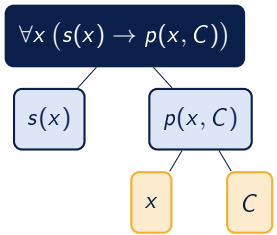
2. Predicate Formulae and wffs

General atomic formula: $p(a_1, a_2, \ldots, a_n)$

A **term** a_i is a constant, a variable, or a function.

Well-formed formulae

1. Atomic formula is a wff
2. p wff $\Rightarrow \neg p$, (p) wff
3. p, q wff $\Rightarrow p \wedge q,\ p \vee q,\ p \rightarrow q,\ p \leftrightarrow q$
4. p wff $\Rightarrow \exists x\, p(x),\ \forall x\, p(x)$ wff
5. Nothing else is a wff



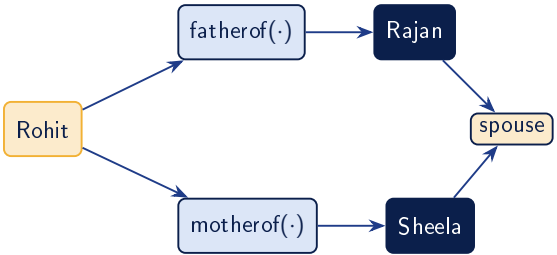
two-place predicate: "x plays Cricket"

3. Functions and Arity

- A function returns an object; a predicate returns true/false.
- Arity = number of arguments.
- Argument **order matters**.

Examples

$plus(2, 2) = 4$
$square(5) = 25$
$product(5, 10, 15) = 750$



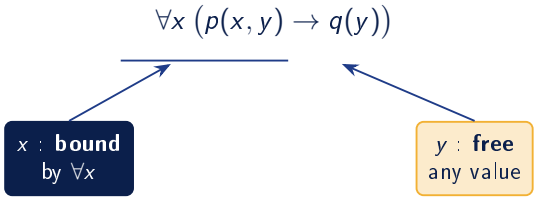
spouse(fatherof(Rohit), motherof(Rohit))

4. Quantifiers, Bound and Free Variables

- $\forall$ "for all" $\quad \exists$ "there exists"
- All officers pay tax: $\forall x\,(o(x) \to p(x))$
- Some persons are clever: $\exists x\,(person(x) \wedge c(x))$
- Multiple quantifiers: $\forall x\,\forall y\,g(x,y)$

Note

For $\exists$, the connective is normally $\wedge$, not $\to$ (the book writes $\to$ in some examples).



In $\forall x(p(x)) \to q(x)$ the quantifier covers only $p(x)$; rename: $\forall x(p(x)) \to q(y)$.

5. Inference Rules for Quantifiers

Universal Instantiation

$$\frac{\forall x\, p(x)}{\therefore\ p(a)},\quad a \in U$$

Universal Generalization

$$\frac{p(a) \text{ for all } a \in U}{\therefore\ \forall x\, p(x)}$$

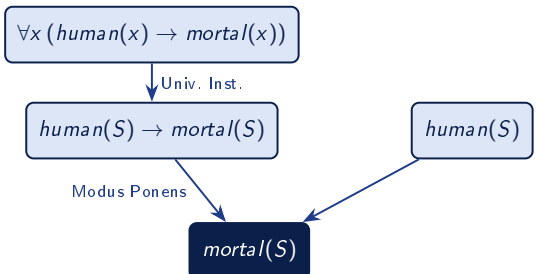
Existential Instantiation

$$\frac{\exists x\, p(x)}{\therefore\ p(a) \text{ for some } a}$$

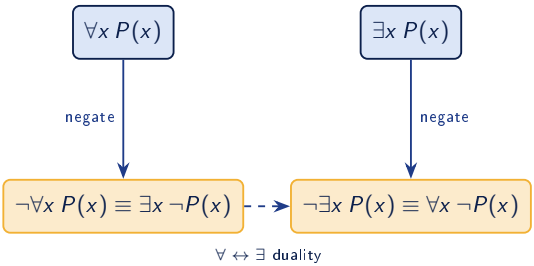
Existential Generalization

$$\frac{p(a) \text{ for some } a}{\therefore\ \exists x\, p(x)}$$

Socrates example



6. Negating Quantifiers and Domain of Discourse



Wild animals: "Not all are dangerous" $\equiv$ "some wild animal is not dangerous".

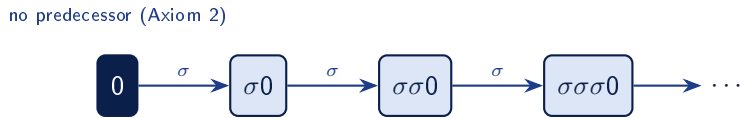
Domain of discourse (universe) restricts the variables:

- Unrestricted: $\forall x\,(p(x) \to \exists y\,(p(y) \wedge g(y,x)))$
- Domain = positive integers: $\forall x\, \exists y\, g(y,x)$

"$x > 10$" over domains

Domain	$\forall x$	$\exists x$
$\{20, 30, 40\}$	T	T
$\{5, 10, 15\}$	F	T
$\{1, 2, 3, 4\}$	F	F

7. Peano Axioms: Building the Naturals



1. $\mathbb{N}$ is a set, $0 \in \mathbb{N}$, $\sigma : \mathbb{N} \to \mathbb{N}$
2. $\neg\exists n\,(\sigma n = 0)$
3. σ injective: $\sigma x = \sigma y \Rightarrow x = y$
4. **Induction**: $S \subseteq \mathbb{N},\ 0 \in S,\ n \in S \Rightarrow \sigma n \in S$ gives $S = \mathbb{N}$

Iteration: $f^0 = \mathrm{id},\ f^{\sigma n} = f \circ f^n$

Addition: $m + n = \sigma^n(m)$

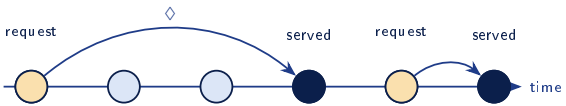
Proved: $m + 0 = m,\ 1 + n = \sigma n$, and commutativity $x + y = y + x$ (by induction).

Multiplication and exponentiation follow by iterating addition and multiplication.

Higher-order logic
Quantifies over predicates themselves (e.g. over "tastes"), unlike first-order logic that quantifies over objects; limited practical use.

Temporal logic
Adds time-dependent operators:

- $\Box P$: always
- $\Diamond P$: eventually
- $\Box\Diamond P$: again and again
- $\Box(P \Rightarrow \Diamond Q)$: leads-to



Live service: $\Box(\mathit{requested} \Rightarrow \Diamond\, \mathit{served})$

Also: termination $\Diamond\, \mathit{terminated}$, mutual exclusion $\Box P$, model checking

Reference

Source of this presentation

Prof. K.R. Chowdhary, *Fundamentals of Discrete Mathematical Structures*, PHI Learning.
ISBN 978-81-203-5074-8

Book link

https://www.phindia.com/Books/BookDetail/9788120350748/fundamentals-of-discrete-mathematical-structures-chowdhary