Mathematical Induction, Recursion and Recurrence Relations

Fundamentals of Discrete Mathematical Structures

Based on the book by Prof. K.R. Chowdhary

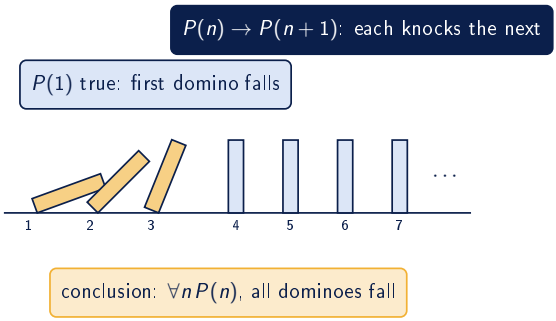
$$P(0) \wedge (P(n) \rightarrow P(n+1)) \;\Rightarrow\; \forall n\, P(n)$$

1. The Idea of Mathematical Induction

- A small sample plus a rule for the unexamined cases gives a conclusion about every case, with deductive certainty.
- To prove $\forall n\, P(n)$:
  1. **Basis step**: show $P(0)$ (or the starting value).
  2. **Induction step**: assume $P(n)$ (the *hypothesis*), show $P(n+1)$.

Rule of inference

$[P(0) \wedge (P(n) \rightarrow P(n+1))] \rightarrow \forall n\, P(n)$

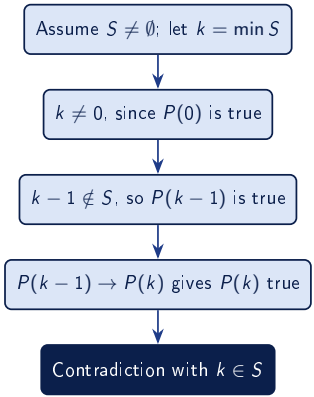


2. Why Induction Works: Well-Ordering

Suppose $P(0)$ and $P(n) \to P(n+1)$ hold, yet $P(n)$ fails somewhere.

Then $S = \{n : P(n) \text{ false}\}$ is non-empty, so by the **well-ordering property** it has a least element k.

This leads to a contradiction, so $S = \emptyset$ and $P(n)$ holds for all n.



3. Worked Proofs: Sums and Inequalities

Claim: $1 + 3 + \cdots + (2n-1) = n^2$.
Basis: $n = 1$: $1 = 1^2$.
Step: assume it holds for n. Then

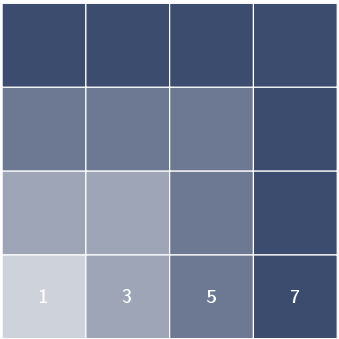
$$\underbrace{1 + \cdots + (2n-1)}_{n^2} + (2n+1) = n^2 + 2n + 1 = (n+1)^2.$$

Claim: $n < 2^n$ for all $n \geq 1$.
Basis: $1 < 2^1$.
Step: from $n < 2^n$, $n + 1 < 2^n + 1 \leq 2^n + 2^n = 2^{n+1}$.

Starting value

The basis need not be 0; it can be any value, with the result then proved upward from it.

$$1 + 3 + 5 + 7 = 4^2$$



layers $1, 3, 5, 7$ build a 4×4 square

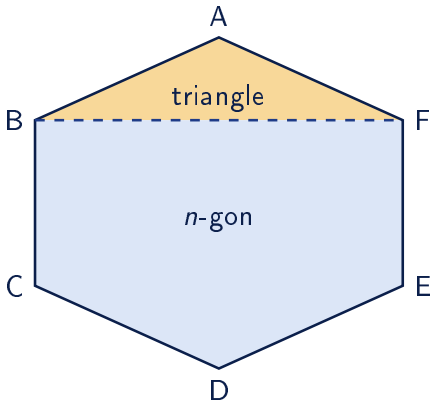
4. Applications: Polygon Angles and Factors

Polygon: for $n \geq 3$ the interior angles of an n-gon sum to $(n-2) \cdot 180^\circ$.
Basis: $n = 3$: 180°.
Step: an $(n+1)$-gon is an n-gon plus a triangle:

$$(n-2)180^\circ + 180^\circ = (n-1)180^\circ.$$

Factor: $a - b$ divides $a^n - b^n$.
Hypothesis: $a^k - b^k = (a-b)M$.

$$\begin{aligned} a^{k+1} - b^{k+1} &= (a-b)a^k + b(a^k - b^k) \\ &= (a-b)(a^k + bM) \end{aligned}$$

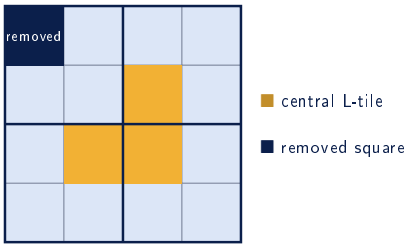


5. Application: Tiling with L-shaped Tiles

Claim: a $2^n \times 2^n$ board with one square removed can be tiled by L-shaped tiles (3 squares each).

Basis ($n = 1$): a 2×2 board minus one square is exactly one L-tile.

Step: split a $2^{n+1} \times 2^{n+1}$ board into four $2^n \times 2^n$ quadrants. Place one L-tile at the centre covering one square of each of the three quadrants that lack the removed square. Every quadrant now has exactly one square missing, so the hypothesis tiles each.

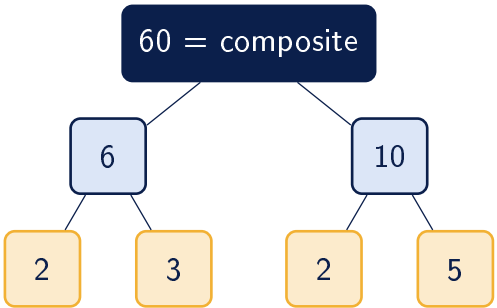


6. Second Principle (Strong) Induction

Assume $P(0), P(1), \ldots, P(n)$ all hold, then prove $P(n+1)$:

$$\big[P(0) \wedge P(1) \wedge \cdots \wedge P(n)\big] \to P(n+1).$$

Claim: every integer $n > 1$ is a product of primes.
Basis: 2 is prime.
Step: if $n+1$ is prime we are done. Otherwise $n+1 = a \cdot b$ with $2 \le a, b \le n$. By hypothesis a and b are products of primes, so $n+1$ is too.



$60 = 2 \cdot 3 \cdot 2 \cdot 5$, using $P(6)$ and $P(10)$

7. Recursion and the Towers of Hanoi

Recursive definition: base values + a rule for the rest.

$fact(0) = 1, \quad fact(n) = n \cdot fact(n-1)$
$fact(3) = 3 \cdot 2 \cdot 1 \cdot 1 = 6$

Hanoi: let $M(n)$ be the number of moves.

$$M(0) = 0, \qquad M(n) = 2M(n-1) + 1$$

n	1	2	3	4	5
$M(n)$	1	3	7	15	31

Closed form: $M(n) = 2^n - 1$ (provable by induction).



1. move top $n-1$ disks: peg 1 $\rightarrow$ peg 2
2. move disk n: peg 1 $\rightarrow$ peg 3
3. move $n-1$ disks: peg 2 $\rightarrow$ peg 3

$$M(n) = M(n-1) + 1 + M(n-1)$$

8. Recurrences, Iteration and Recursion Trees

First-order linear: $T(n) = rT(n-1) + a$. Iterating down to $T(0) = b$:

$$T(n) = r^n b + a \sum_{i=0}^{n-1} r^i.$$

Linear recurrences:

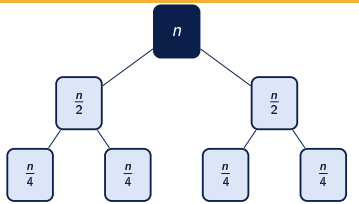
- Fibonacci: $F(n+2) = F(n+1) + F(n)$, degree 2
- Hanoi: $h_n = 2h_{n-1} + 1$, degree 1, non-homogeneous

Divide and conquer (merge sort):

$$T(n) = 2T(n/2) + n, \quad T(1) = 1$$

Iterating: $T(n) = 2^i T(n/2^i) + i\,n$. At $i = \log_2 n$:

$$T(n) = n \log_2 n + n.$$



Reference

Source of this presentation

Prof. K.R. Chowdhary, *Fundamentals of Discrete Mathematical Structures*, PHI Learning.
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Book link

https://www.phindia.com/Books/BookDetail/9788120350748/fundamentals-of-discrete-mathematical-structures-chowdhary