Combinatorics

Fundamentals of Discrete Mathematical Structures

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$$^{n}C_{k} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \quad \Rightarrow \quad 2^{n} = \sum_{k=0}^{n} \binom{n}{k}$$

Counting: Lists, Sets and Multisets

- **List:** ordered collection (string, sequence, word); order matters.
- **Set:** collection of distinct objects; order does not matter.
- **Multiset:** repeats allowed, order does not matter.
- $|A|$ denotes the number of elements (cardinality) of A.

Example

For $S = \{x, y, z\}$ there are 6 lists without repetition: $xyz, xzy, yxz, yzx, zxy, zyx$ (the permutations of S).

- Ram buys {3 bananas, 2 apples} and Rajni buys {2 apples, 3 bananas}: same multiset, since only the counts matter.

Rules of Sum and Product

Rule of Sum

If task t_1 can be done in n_1 ways, t_2 in n_2 ways, and they cannot be done together, then either can be done in $n_1 + n_2$ ways. For disjoint sets:
$|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|$

Rule of Product

If step 1 can be done in n_1 ways and step 2 in n_2 ways, the whole task can be done in $n_1 \times n_2$ ways.

- Textbooks: $15 + 5 = 20$ ways to read one book.
- Password (2 letters + 4 digits): $26 \times 26 \times 10^4 = 6{,}760{,}000$.
- Bit strings of length 8: $2^8 = 256$.
- Sum 8 with two dice: 5 outcomes if distinguishable $((2,6),(6,2),(3,5),(5,3),(4,4))$, only 3 if not $((2,6),(3,5),(4,4))$.

Power Set

- The power set of A is $P(A) = \{X \mid X \subseteq A\}$.
- For $\{a, b, c\}$: $\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$, giving 8 subsets.

Theorem

A set with n elements has 2^n subsets.

- **Reason:** for each element there are two decisions (include or exclude), and the choices multiply: $2 \times 2 \times \cdots \times 2 = 2^n$.
- Binomial form: $2^n = \sum_{r=0}^{n} \binom{n}{r}$, the number of ways of choosing $0, 1, \ldots, n$ objects.

Principle of Inclusion and Exclusion

Two and three sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Generalized (proved by induction)

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

Example: 30 students; games A, B, C played by $10, 20, 12$; all three by 5; $|B \cap C| = 7$, $|A \cap C| = 8$. Then $30 = 32 - |A \cap B|$, so $|A \cap B| = 2$.

Pigeonhole Principle

Basic form

If p pigeons are placed in h holes and $p > h$, some hole contains more than one pigeon.

- **Extended:** if $p > nh$, some hole holds at least $n + 1$ pigeons.
- **Generalized:** n objects in m containers gives some container with at least $\lceil n/m \rceil$ objects.
- **Strong form:** if $q_1 + \cdots + q_n - (n - 1)$ objects go into n boxes, then box i holds at least q_i objects for some i.
- **Uses:** hash-table collisions; no compression algorithm can shrink every file.

Examples: in a group of n people, two have the same number of friends. From $\{1, \ldots, 100\}$ any 51 chosen integers contain two where one divides the other.

Injection, Surjection and Bijection

- **Injective (one-to-one):** $f(a) = f(b) \Rightarrow a = b$.
- **Surjective (onto):** for every $b \in B$ there is $a \in A$ with $b = g(a)$.
- **Bijective:** both injective and surjective.
- **Permutation:** a bijection $h : A \to A$ of a set onto itself.

Comparing set sizes

For infinite sets, a bijection shows equal cardinality. For example, $f : \mathbb{N} \to 2\mathbb{N}$ and squaring $f : \mathbb{N} \to \mathbb{N}^2$ are bijections, so these sets have the same size as $\mathbb{N}$ even though they are properly contained in it.

Permutations

- A permutation of n elements is an arrangement in which order matters. The number is $n! = n(n-1)!$, with $0! = 1$.
- Permutations of size k from n objects:

$$^nP_k = \frac{n!}{(n-k)!}$$

- Word CAT, two-letter groups: $^3P_2 = \frac{3!}{1!} = 6$: CA, CT, AC, AT, TC, TA.
- Four-letter words from 10 letters: $^{10}P_4 = 10 \times 9 \times 8 \times 7 = 5040$.
- Words of any length from SQUARE: $^6P_1 + \cdots + {}^6P_6 = 1956$.
- A *derangement* is a permutation in which no object stays in its natural place.

Combinations and Pascal's Triangle

Combinations (order does not matter)

$$^{n}C_{k} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{^{n}P_{k}}{k!}$$

- $^{3}C_{2} = 3$: CA, CT, AT.
- $^{10}C_{4} = \frac{5040}{4!} = 210$.
- Recurrence: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
- Triangular numbers: $C(n+1, 2) = \frac{n(n+1)}{2}$.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & 6 & & 15 & & 20 & & 15 & & 6 & \\
\end{array}$$

Reference

Source of this presentation

Prof. K.R. Chowdhary, *Fundamentals of Discrete Mathematical Structures*, PHI Learning.
ISBN 978-81-203-5074-8

Book link

https://www.phindia.com/Books/BookDetail/9788120350748/FUNDAMENTALS-OF-DISCRETE-MATHEMATICAL-STRUCTURES-CHOWDHARY