Discrete Probability

Fundamentals of Discrete Mathematical Structures

by Prof K R Chowdhary

$$P(A \mid B) = \frac{P(AB)}{P(B)} \quad \Rightarrow \quad P(B_j \mid A) = \frac{P(A \mid B_j)P(B_j)}{\sum_i P(A \mid B_i)P(B_i)}$$

Events and Their Relations

- Under a set of conditions Ψ, an event is *certain* (always occurs), *impossible* (never occurs) or *random* (may or may not occur).
- $A \subset B$: A implies B. $A = B$: each implies the other.
- **Product** $AB = A \cap B$: both occur. **Sum** $A + B = A \cup B$: at least one occurs. **Difference** $A - B$: A occurs but B does not.
- **Contrary events:** $A + \bar{A} = S$ and $A\bar{A} = \emptyset$ (null).

Example (one die)

A: six, B: three, C: even, D: multiple of three. Then $A \subset C$, $A \subset D$, $B \subset D$, $A + B = D$ and $CD = A$.

Sample Space and Distribution Function

- The **sample space** $S = \{X \mid X \text{ is a possible outcome}\}$ is the set of all outcomes; it is *discrete* if finite or countably infinite.

Distribution function $m : S \to \mathbb{R}$

(1) $m(\omega) \geq 0$ for all $\omega \in S$, and (2) $\sum_{\omega \in S} m(\omega) = 1$.

For $E \subseteq S$: $P(E) = \sum_{\omega \in E} m(\omega)$.

- For disjoint A, B: $P(A \cup B) = P(A) + P(B)$.
- Inclusion–exclusion carries over:
  $P\big(\bigcup A_i\big) = \sum P(A_i) - \sum P(A_i A_j) + \cdots + (-1)^{n-1} P(A_1 \cdots A_n)$.
- **Two coins:** $m(HH) + m(HT) + m(TH) + m(TT) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$.

Classical Definition of Probability

Definition

If A decomposes into m cases of a complete group of n mutually exclusive, equally probable events, then $P(A) = \dfrac{m}{n}$.

- One die, $C = E_2 + E_4 + E_6$ (even number): $P(C) = \frac{3}{6} = \frac{1}{2}$; $P(E_1 + E_2) = \frac{1}{3}$.
- Two dice (36 equally likely outcomes):

Sum	2	3	4	5	6	7	8	9	10	11	12
P	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

- A roll of 11 can occur in two ways, so $P = \frac{2}{36} = \frac{1}{18}$.

Basic Properties of Probability

Theorem (addition of probabilities)

1. $P(A) \geq 0$ for every event A.
2. $P(U) = 1$ for the certain event U.
3. If A splits into mutually exclusive cases B and C, then $P(A) = P(B) + P(C)$.

Theorem (complement)

$P(\bar{A}) = 1 - P(A)$.

- **Proof idea (addition):** if B is favoured by m' cases and C by m'' cases, then $P(A) = \frac{m'+m''}{n} = \frac{m'}{n} + \frac{m''}{n}$.
- **Proof idea (complement):** $A + \bar{A} = U$ and the two are exclusive, so $P(A) + P(\bar{A}) = 1$.

Worked Examples with Cards

Three cards from a deck of 36 (4 aces): exactly one ace

$$P = \frac{\binom{4}{1}\binom{32}{2}}{\binom{36}{3}} = \frac{4 \cdot 496}{7140} \approx 0.278$$

Three cards from 52: at least one ace

$$P(A_1) = \frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}} \approx 0.2041, \quad P(A_2) = \frac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}} \approx 0.0130, \quad P(A_3) = \frac{\binom{4}{3}\binom{48}{0}}{\binom{52}{3}} \approx 0.00018$$

$$P(A) = P(A_1) + P(A_2) + P(A_3) = 0.21728 \ (21.728\%)$$

Frequency versus Probability

- The classical definition fails when equally likely cases cannot be separated (for example, the chance a newborn is a boy).
- If event A occurs μ times in n independent trials, the **frequency** μ/n becomes nearly constant as n grows.
- Frequency has three properties: it is 1 for a certain event, 0 for an impossible event, and for mutually exclusive $A_1, \ldots, A_n$ the frequency of their sum is the sum of the frequencies.

Statistical definition (R. von Mises)

$$P = \lim_{n \to \infty} \frac{\mu}{n}$$

Probability must be 1 for a certain event, 0 for an impossible one, and additive over mutually exclusive events: $P(C) = P(A_1) + \cdots + P(A_n)$.

Conditional Probability

Definition

Given that B has occurred (with $P(B) > 0$),

$$P(A \mid B) = \frac{P(AB)}{P(B)}, \qquad P(B \mid A) = \frac{P(AB)}{P(A)}$$

Two dice, sum 8 given an even sum: $P(\text{sum } 8) = \frac{5}{36}$, but even sums have 18 outcomes, so $P(A \mid B) = \frac{5}{18}$.

Two cards, second is an ace:

- Unconditional: $P(A) = \frac{4 \cdot 3}{52 \cdot 51} + \frac{48 \cdot 4}{52 \cdot 51} = \frac{1}{13}$.
- Given the first card was an ace: $P(A \mid B) = \frac{3}{51}$.

Independence, Total Probability and Bayes' Rule

- **Multiplication:** $P(AB) = P(A)P(B \mid A) = P(B)P(A \mid B)$.
- **Independent** events: $P(A \mid B) = P(A)$, hence $P(AB) = P(A)P(B)$.
- Events $B_1, \ldots, B_m$ are *collectively independent* if every sub-collection satisfies $P(B_{i_1} \cdots B_{i_r}) = P(B_{i_1}) \cdots P(B_{i_r})$.

Total probability (for exclusive, exhaustive $B_1, \ldots, B_n$)

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i)\, P(B_i)$$

Bayes' rule

$$P(B_j \mid A) = \frac{P(A \mid B_j)\, P(B_j)}{\sum_i P(A \mid B_i)\, P(B_i)}$$

$P(B_j)$ is the prior probability and $P(B_j \mid A)$ the posterior probability.

Reference

Source of this presentation

Prof. K.R. Chowdhary, *Fundamentals of Discrete Mathematical Structures*, PHI Learning.
ISBN 978-81-203-5074-8

Book link

https://www.phindia.com/Books/BookDetail/9788120350748/FUNDAMENTALS-OF-DISCRETE-MATHEMATICAL-STRUCTURES-CHOWDHARY