Graph Theory and Graph Algorithms

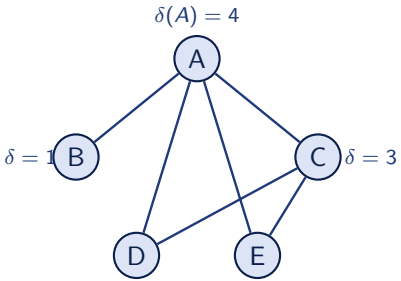
Fundamentals of Discrete Mathematical Structures

by Prof K R Chowdhary

$$G = (V, E), \qquad \sum_{v \in V} \delta(v) = 2|E|, \qquad d(a, C) = \min_{u \in C,\, v \notin C} \{d(a, u) + wt(u, v)\}$$

Graphs and Terminology

- A graph is $G = (V, E)$: vertices V and edges $E = \{(u, v) \mid u, v \in V\}$.
- **Simple** (no loops), **complete** K_n (clique), **directed**, **weighted**, **multigraph**.
- **Connected** if every pair is joined; otherwise it splits into *connected components*.
- **Trail:** no repeated edge. **Path:** no repeated internal vertex.
- Distance $d(u, v)$ = edges on a shortest path; *diameter* = max distance.
- **Complement** $\bar{G}$: same vertices, edges exactly where G has none.



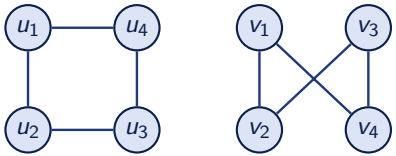
Five scientists: an edge means "know each other"

Degrees and Isomorphism

Theorem 8.1 and 8.2

In every graph the number of odd-degree vertices is even, and $\sum_{v \in V} \delta(v) = 2|E|$. For digraphs, $\delta(v) = \delta^-(v) + \delta^+(v)$.

- G and G' are **isomorphic** if a map $f : V \to V'$ is (i) one-to-one, (ii) onto, and (iii) $\{u, v\} \in E \iff \{f(u), f(v)\} \in E'$.
- Necessary: same $|V|$, same $|E|$. Test adjacency by comparing the matrices M_G and $M_{G'}$.
- An **automorphism** is an isomorphism of a graph onto itself.



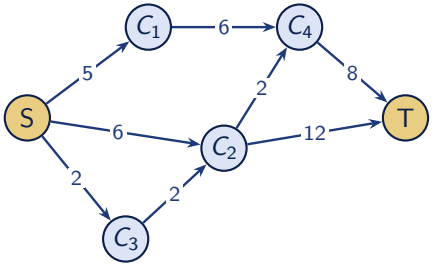
$u_1 \mapsto v_1,\ u_2 \mapsto v_2,\ u_3 \mapsto v_3,\ u_4 \mapsto v_4$

Dijkstra's Shortest Path Algorithm

- Grow a **core set** C starting from $\{a\}$; label each vertex $(L(v), \text{pred})$.
- Each step, for $v \notin C$: $L(v) = \min_{u \in C}\{L(v),\ L(u) + wt(u, v)\}$
- Move the vertex with the smallest label into C; stop when the target joins.

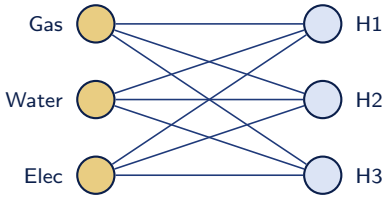
Core adds	S	C_3	C_2	C_1	C_4	T
Time	0	2	4	5	6	14

Route: $S \to C_3 \to C_2 \to C_4 \to T$ (14 hours).



Planar Graphs

- A graph is **planar** if it can be drawn in the plane with edges meeting only at vertices.
- **Utilities puzzle:** join gas, water and electricity to three houses with no crossing. Any eight of the nine connections can be laid without crossing, but not all nine.
- The graph is therefore **nonplanar**: the complete bipartite graph $K_{3,3}$.
- Kuratowski's K_5 is also nonplanar. A graph that looks nonplanar may be redrawn: the cube graph is planar.



$K_{3,3}$: each supply meets each house

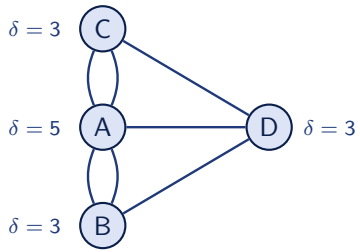
Eulerian Graphs and the Königsberg Bridges

- An **Eulerian circuit** is a closed walk that uses every edge exactly once.

Euler's theorem

A connected graph has an Eulerian circuit iff every vertex has even degree. It has an Eulerian path iff exactly two vertices have odd degree.

- Königsberg: $\delta(A) = 5$ and $\delta(B) = \delta(C) = \delta(D) = 3$, all odd, so the seven-bridge stroll is impossible.
- Reason: a walk enters and leaves each middle vertex, using edges in pairs.



Multigraph of the seven bridges

Hamiltonian Graphs and the Travelling Salesman

- A **Hamiltonian circuit** visits every *vertex* exactly once and returns to the start. Deciding if one exists is **NP-complete**.
- **Dirac (1952):** a graph with $n \geq 3$ and minimum degree $\delta \geq n/2$ is Hamiltonian. More generally, $\deg u + \deg v \geq n$ for every non-adjacent pair suffices.
- **TSP:** find the cheapest tour through all n cities. It needs checking $\frac{(n-1)!}{2}$ circuits, which is a combinatorial explosion: for $n = 20$ there are over 1.2×10^{17} paths.

Dynamic programming (principle of optimality)

Let $f(i; j_1, \ldots, j_k)$ be the shortest path from i through the unvisited cities to city 0. Then

$$f(i; j_1, \ldots, j_k) = \min_{1 \leq m \leq k} \{ d_{ij_m} + f(j_m; j_1, \ldots, j_{m-1}, j_{m+1}, \ldots, j_k) \}, \qquad f(i; j) = d_{ij} + d_{j0}$$

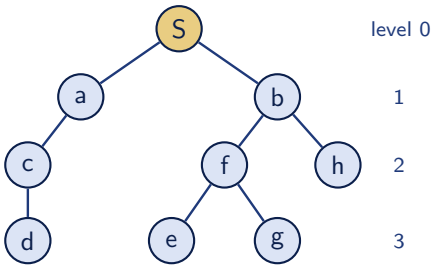
Graph Search: BFS and DFS

- **BFS:** visit all vertices at distance i from S before distance $i+1$ (queue-like, level by level). It generalises Dijkstra with unit edge lengths.
- **DFS:** go as deep as possible first, recursively, then backtrack.

BFS order: $S, a, b, c, f, h, d, e, g$

DFS order: $S, a, c, d, e, f, g, h, b$

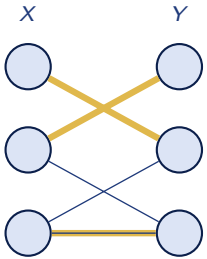
Each BFS level equals the edge-distance from S.



BFS tree rooted at S

Colouring, Bipartite Graphs and Matchings

- **Proper k-colouring:** $\alpha : V \to \{1, \dots, k\}$ with $\alpha(u) \neq \alpha(v)$ for every edge. The **chromatic number** $\chi(G)$ is the least such k (NP-complete to find).
- **Greedy colouring:** give each vertex the lowest legal colour; it never recolours (used in timetables and register allocation).
- **Bipartite:** $V = X \cup Y$ with edges only between X and Y. Bipartite $\iff$ 2-colourable $\iff$ no odd cycle.
- **Matching:** a set of edges with no shared vertex. It is **maximum** if no larger one exists and **perfect** if it saturates every vertex.



Bipartite graph with a perfect matching (gold)

Reference

Source of this presentation

Prof. K.R. Chowdhary, *Fundamentals of Discrete Mathematical Structures*, PHI Learning, Chapter 8: Graph Theory.
ISBN 978-81-203-5074-8

- Euler (1736) founded graph theory with the Königsberg bridge problem; Hamilton (1856) introduced the Icosian game.
- Dijkstra published his shortest path algorithm in 1959.

Book link

https://www.phindia.com/Books/BookDetail/9788120350748/FUNDAMENTALS-OF-DISCRETE-MATHEMATICAL-STRUCTURES-CHOWDHARY