Relations

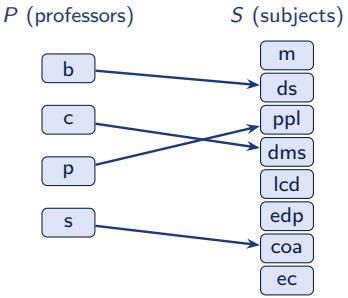
Fundamentals of Discrete Mathematical Structures

by Prof K R Chowdhary

$R \subseteq A \times B, \qquad M_{R_1 \circ R_2} = M_{R_1} \odot M_{R_2}, \qquad R \text{ transitive} \iff R^n \subseteq R$

Relations on Sets

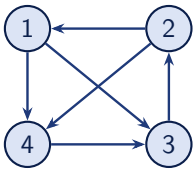
- **Cartesian product:** $A \times B = \{(a, b) \mid a \in A,\ b \in B\}$.
- A **binary relation** from A to B is any subset $R \subseteq A \times B$. Write $a\,R\,b$ if $(a, b) \in R$.
- **Domain** $Dom(R)$: elements of A that are related; **range** $Ran(R)$: the related elements of B.
- **Counting:** $|A \times B| = |A||B|$, so there are $2^{|A||B|}$ relations from A to B, and 2^{n^2} on a set of n elements.
- A relation with $(a, b), (a, c) \in R \Rightarrow b = c$ is a **function** $f : A \to B$. It is **one-to-one** if $a \neq b \Rightarrow f(a) \neq f(b)$.



$R = \{(b, ds), (c, dms), (p, ppl), (s, coa)\}$: who teaches what

Representing Relations

- **Table / matrix:** for $A = \{a_1..a_m\}$, $B = \{b_1..b_n\}$, $M_R = [m_{ij}]$ is $m \times n$ with $m_{ij} = 1$ if $(a_i, b_j) \in R$, else 0.
- **Digraph:** vertices are elements, an arrow $a_i \to a_j$ for each $(a_i, a_j) \in R$.
- **Arrays:** edges numbered $1..e$, with a tail array and a head array. A matrix cell has address $k + ((i-1) \cdot n + (j-1))\, w$.
- **Linked lists:** for each vertex, a list of its neighbours; the list heads sit in a pointer array. Nodes can be added or freed dynamically.
- The matrix is easily rebuilt from the lists: row i has a 1 in each listed column.



1	$\to 3 \to 4$
2	$\to 1 \to 4$
3	$\to 2$
4	$\to 3$

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R = \{(1,3), (1,4), (2,1), (2,4), (3,2), (4,3)\}$$

Combination Relations

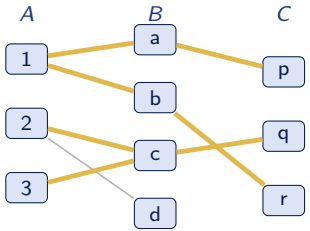
Let $R_1, R_2 \subseteq A \times B$. Since relations are sets, $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$ and $R_1 \oplus R_2$ are relations too. Take $R_1 =$ courses students are *interested in*, $R_2 =$ courses they have *taken*:

Relation	Meaning	Matrix entry m_{ij}
$R_1 \cup R_2$	interested in or taken	$m^1_{ij} \vee m^2_{ij}$
$R_1 \cap R_2$	interested in and taken	$m^1_{ij} \wedge m^2_{ij}$
$R_1 - R_2$	interested in but not taken	$m^1_{ij} = 1,\ m^2_{ij} = 0$
$R_1 \oplus R_2$	in exactly one of the two	$m^1_{ij} \neq m^2_{ij}$

- All four are computed entry by entry on the relation matrices, so a program needs only Boolean operations.
- Different from these is **composition** of relations, next.

Composition and Powers of Relations

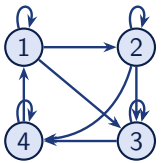
- For $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$: $R_1 \circ R_2 = \{(a, c) \mid \exists b : (a, b) \in R_1,\ (b, c) \in R_2\}$.
- **Matrix form:** $M_{R_1 \circ R_2} = M_{R_1} \odot M_{R_2}$, a Boolean product with $m_{ij} = \bigvee_k (m^1_{ik} \wedge m^2_{kj})$. In a digraph: a path through an intermediate vertex.
- Composition is **associative**: $(R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3)$.
- **Powers** on one set: $R^1 = R$, $R^{n+1} = R^n \circ R$, and $M_{R^n} = M_R^{[n]}$.



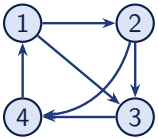
$R_1 \circ R_2 = \{(1, p), (1, r), (2, q), (3, q)\}$ (gold paths)

Reflexive and Irreflexive Relations

- R on A is **reflexive** if $(a, a) \in R$ for every $a \in A$. Example: "a divides b", or "same department".
- **Irreflexive:** $(a, a) \notin R$ for every a. Example: "is married to".
- **Digraph test:** reflexive means a loop at every vertex; irreflexive means no loops.
- **Matrix test:** reflexive means the diagonal is all 1; irreflexive means the diagonal is all 0.
- A relation can be neither: it is not reflexive just because some (a, a) are missing.



reflexive



irreflexive

Same arrows; the left one adds all four loops.

Symmetric, Antisymmetric and Asymmetric

- **Symmetric:** $(a, b) \in R \Rightarrow (b, a) \in R$. Matrix: M_R equals its transpose.
- **Antisymmetric:** $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$.
- **Asymmetric:** $(a, b) \in R \Rightarrow (b, a) \notin R$. So no loops, and every arrow is one-way.
- Symmetric and antisymmetric are *not* opposites: the relation $a = b$ is both.

Relation on $\mathbb{Z}$	Symmetric	Antisymmetric	Asymmetric
$a > b$	no	yes	yes
$a = b$	yes	yes	no
$a + b \leq 5$	yes	no	no

Reading the three tests

For $a > b$ the pair (b, a) is never also present, so it is asymmetric (and hence antisymmetric).
For $a + b \leq 5$, both $(1, 2)$ and $(2, 1)$ occur with $1 \neq 2$, so it fails both.

Closures of Relations

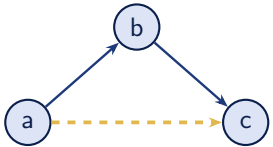
- The **closure** of R for a property P is the smallest relation $R' \supseteq R$ having P (contained in every such relation).
- **Reflexive closure:** $R \cup \Delta$, where $\Delta = \{(a, a) \mid a \in A\}$ is the diagonal relation.
- **Inverse:** $R^{-1} = \{(b, a) \mid (a, b) \in R\}$. In matrix form it is the transpose: $inv(j, i) = m(i, j)$.
- **Symmetric closure:** $R \cup R^{-1}$.
- **Complement:** $\bar{R} = \{(a, b) \in A \times A \mid (a, b) \notin R\}$. This is not the inverse.

Example

$R = \{(1, 2), (1, 1), (2, 3), (3, 2), (4, 1), (4, 2)\}$ on $\{1, 2, 3, 4\}$. Adding the missing mirror pairs $(2, 1), (1, 4), (2, 4)$ gives its symmetric closure.
For $R = \{(a, b) \mid a \geq b\}$ on the integers, $R \cup R^{-1}$ holds for every pair, since any two numbers are comparable.

Transitive Relations

- R is **transitive** if $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$.
- **Digraph:** every two-step path $a \to b \to c$ has a shortcut $a \to c$.
- **Matrix:** whenever $m_{ik} = 1$ and $m_{kj} = 1$, also $m_{ij} = 1$.
- **Theorem:** R is transitive iff $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$ (shown by induction, using $R^{n+1} = R^n \circ R$).
- **Transitive closure:** the smallest transitive relation containing R.



$R = \{(a, b) \mid a \leq b\}$ on $\{1, 2, 3, 4\}$ is transitive:

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reference

Source of this presentation

Prof. K.R. Chowdhary, *Fundamentals of Discrete Mathematical Structures*, PHI Learning, Chapter 9: Relations.
ISBN 978-81-203-5074-8

- Relations underlie sets, functions and the ordering used in induction proofs.
- Gottfried Leibniz (1646–1716), whose ideas on formal reasoning shaped modern logic and discrete mathematics, is the chapter's historical note.

Book link

https://www.phindia.com/Books/BookDetail/9788120350748/FUNDAMENTALS-OF-DISCRETE-MATHEMATICAL-STRUCTURES-CHOWDHARY