Normalization of Context-free Grammars

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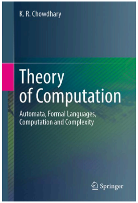

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Introduction to Simplification of Grammars

- Parsing becomes easier if we add full generality in the grammar G.
- In simplification of G, we remove 1) null-productions (also called $\varepsilon-$productions), 2) unit-productions (chain-rules), and 3) non-reachable symbols and corresponding productions.
- But, the generating power of the grammar remains the same.
- A variable symbol is useful if it appears in some derivation, otherwise it is useless.
- if $\varepsilon \notin L(G)$, then all the $\varepsilon-$productions can be removed from the grammar.
- Normal Forms: Is some restrictions in the productions rules, for the right hand side of a production, so that all the productions are in some standard form. Two types normal forms exits:

  CNF: Chomsky Normal Form Grammar: All the productions are like: $A \to BC$, $A \to a$, where $A, B, C \in V$ and $a \in \Sigma$.

  GNF: Greibach Normal Form Grammar: All the productions are like $A \to a\alpha$, where $a \in \Sigma$ and $\alpha \in V^*$.

ε-productions, useless productions

- Productions of the form $A \rightarrow \varepsilon$ are called null productions. All these productions can be eliminated from grammar if $\varepsilon \notin L(G)$. Hence, in this case the generating power of G remains unchanged.
- If $A \Rightarrow^* \varepsilon$ then variable A is called *nullable*.
- If $B \rightarrow X_1 X_2 \dots X_n$, then X_i can be dropped in this production if $\exists X_i \rightarrow \varepsilon$. Such X_i's are called nullable. But if $B \rightarrow \varepsilon$, then X_i's cannot be eliminated.
- **Useless symbols and productions**: Given the derivation $S \Rightarrow^* \alpha X \beta \Rightarrow^* w \in \Sigma^*$, where $\alpha, \beta \in (V \cup \Sigma)^*$, here X is useful because it appears in some derivation.

Example

$S \rightarrow aSb|\varepsilon$, $B \rightarrow bB$, has B useless symbol and $B \rightarrow bB$ as useless production. This is because B is not reachable from S, neither B terminates to null or terminal.

Example

$S \rightarrow A, A \rightarrow aA|\varepsilon, B \rightarrow bA$, has B as useless symbol and $B \rightarrow bA$ as useless production.

Theorem

Theorem

If $G = (V, \Sigma, S, P)$ with $L(G) \neq \phi$, then there is an equivalent grammar $G' = \{V', \Sigma, S, P'\}$, such that G' does not contain useless symbols and productions, and $L(G) = L(G')$.

Proof.

We follow the following algorithm to systematically construct G':

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1. For $A_i \rightarrow u$,, where $u \in \Sigma^*$, move $A_i$ to $V'$, and $A_i \rightarrow u$ to $P'$.
2. $V = V - \{A_i\}, P = P - \{A_i \rightarrow u\}$.
3. while $\exists A_j \rightarrow X_1X_2 \ldots X_n \in P$, where $X_i \in V'$, or $X_i \in \Sigma$ do
   a. $V' = V' \cup \{A_j\}$,
   b. $P' = P' \cup \{A_j \rightarrow X_1X_2 \ldots X_n\}$,
   c. $V = V - \{A_j\}$, $P = P - \{A_j \rightarrow X_1X_2 \ldots X_n\}$
   enddo
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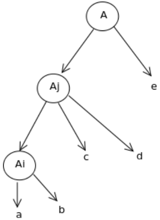
The tree for above is shown in next slide. The grammar constructed as G' does not contains null productions as the same were removed in advance, it does not contain the useless symbols and productions. □

Theorem condinued . . .



The figure shows that first of all we construct the bottom most subtree, A_i, then A_j, and then A, till finally to S. This way only reachable symbols from S, and corresponding productions are move to grammar G'. Therefore, no null or useless productions or symbols have been moved into the grammar G'. Thus generating power of G and G' are same.

Removing unit productions

- Productions of the form $A \rightarrow B$, where $A, B \in V$ are called unit productions.
- Having removed useless symbols and productions, and $\varepsilon-$productions, we remove the unit-productions.
- **Example:** Given productions $A \rightarrow B, B \rightarrow bB|c$, it can be substituted by single production:

  $A \rightarrow bB|c$. □

- If there is sequence of unit productions, and it gives a look of chain like: $A \Rightarrow^* B$, then all the unit productions can be removed systematically. But, if there is situation like: $A \Rightarrow^* B$, due to $A \Rightarrow BC \Rightarrow B$, and $C \rightarrow \varepsilon$, then $A \Rightarrow^* B$ is not a chain.

CNF Theorem

Theorem

If there is grammar $G = (V, \Sigma, S, P$, there exists an equivalent grammar $G' = (V', \Sigma', S, P')$ without unit-productions, $\varepsilon-$productions, and useless symbols and productions, and is CNF.

Proof.

- The $\varepsilon-$productions, unit-productions, and useless-productions and symbols can be removed using the mathods discussed earlier. This does not effect the generating power of the grammar.
- The grammar availabe now has following format: $A \rightarrow X_1X_2 \dots X_n$. For $n = 1$, the right hand side of a production has single symbol. This will be termal only, as all the unit productions have been removed.
- For $X \geq 2$, there is $X_i \in (V \cup \Sigma)$. If X_i is terminal, say a, then substitute a by C_i and introduce a new production $C_i \rightarrow a$.

□

CNF Theorem . . .

- All the productions are now of the form: $A \to a$ or $A \to C_1 C_2 \dots C_n$. All the productions like $A \to C_1 C_2 \dots C_n$ are reduced to $A \to BC$ as follows: ($A \to C_1 C_2$ is already CNF). For $n > 2$, modify the productions as follows:

$$\begin{aligned} A &\to C_1 D_1 \\ D_1 &\to C_2 D_2 \\ &\dots \\ D_{n-3} &\to C_{n-2} D_{n-2} \\ D_{n-2} &\to C_{n-1} D_n. \end{aligned}$$

This proves the theorem. □

Bibliography

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